DEFORMING A LIE ALGEBRA BY MEANS OF A TWO FORM

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Abstract. We consider a vector space V over $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, equipped with a skew symmetric bracket $[\cdot,\cdot] : V \times V \to V$ and a 2-form $\omega : V \times V \to \mathbb{K}$. A simple change of the Jacobi identity to the form $[A,[B,C]] + [C,[A,B]] + [B,[C,A]] = \omega(B,C)A + \omega(A,B)C + \omega(C,A)B$ opens new possibilities, which shed new light on the Bianchi classification of 3-dimensional Lie algebras.

1. Introduction

In reference [2] we considered a real vector space V of dimension n equipped with a Riemannian metric g and a symmetric 3-tensor Υ_{ijk} such that: i) $\Upsilon_{ijk} = \Upsilon_{(ijk)}$, ii) $\Upsilon_{ijj} = 0$ and iii) $\Upsilon_{jki}\Upsilon_{lmi} + \Upsilon_{lji}\Upsilon_{kmi} + \Upsilon_{kli}\Upsilon_{jmi} = g_{jk}g_{lm} + g_{lj}g_{km} + g_{kl}g_{jm}$. Such tensor defines a bilinear product $\{\cdot,\cdot\} : V \times V \to V$ given by

$$\{A,B\}_i = \Upsilon_{ijk}A_jB_k.$$

This product is symmetric

$$\{A,B\} = \{B,A\} \tag{1.1}$$

due to property ii), and it satisfies a three-linear identity:

$$\{A,\{B,C\}\}+\{C,\{A,B\}\}+\{B,\{C,A\}\} = g(B,C)A+g(A,B)C+g(C,A)B, \tag{1.2}$$

due to property iii). Restricting our attention to structures $(V, g, \{\cdot,\cdot\})$ associated with tensors Υ as above, we note that they are related to the isoparametric hypersurfaces in spheres [3, 4]. Using Cartan's results [5] on isoparametric hypersurfaces we concluded in [6] that structures $(V, g, \{\cdot,\cdot\})$ exist only in dimensions $5, 8, 14$ and 26.

A striking feature of property (1.2) is that it resembles very much the Jacobi identity satisfied by every Lie algebra. The main difference is that for a Lie algebra the bracket $\{\cdot,\cdot\}$ should be *anti*-symmetric and that the analog of (1.2) should have r.h.s equal to *zero*.

Adapting properties (1.1)-(1.2) to the notion of a Lie algebra we are led to the following structure.

Definition 1.1. A vector space V equipped with a bilinear bracket $[\cdot,\cdot] : V \times V \to V$ and a 2-form $\omega : V \times V \to \mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ such that

$$[A,B] = -[B,A] \qquad \text{and}$$

$$[A,[B,C]] + [C,[A,B]] + [B,[C,A]] = \omega(B,C)A + \omega(A,B)C + \omega(C,A)B \tag{1.3}$$

is called an ω-deformed Lie algebra.

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This definition obviously generalizes the notion of a Lie algebra and coincides with it when $\omega \equiv 0$. Note also that if the dimension of V is $\dim V = 2$, then $\omega(B,C)A+\omega(A,B)C+\omega(C,A)B \equiv 0$ for any 2-form ω and every $A,B,C \in V$. Thus in 2-dimensions it is impossible to ω-deform the Jacobi identity, and 2-dimensional ω-deformed Lie algebras are just the Lie algebras equipped with a 2-form ω. This is not anymore true if $\dim V \neq 2$. Indeed assuming that $\dim V \neq 2$ and that $\omega(B,C)A+\omega(A,B)C+\omega(C,A)B \equiv 0$ for all $A,B,C \in V$ we easily prove that $\omega \equiv 0$.

The aim of this note is to show that there exist ω-deformed Lie algebras in dimensions greater than 2 which are not just the Lie algebras.

2. Dimension 3.

It follows that if $\dim V \leq 2$ then all the ω-deformed Lie algebras are just the Lie algebras. To show that in $\dim V = 3$ the situation is different we follow the procedure used in the Bianchi classification [1] of 3-dimensional Lie algebras.

Let $\{e_i\}$, $i = 1,2,3$, be a basis of an ω-deformed 3-dimensional Lie algebra. Then, due to skew-symmetry, we have $[e_i,e_j] = c^k_{~ij}e_k$, $\omega(e_i,e_j) = \omega_{ij}$, where $c^k_{~ij} = -c^k_{~ji}$ and $\omega_{ij} = -\omega_{ji}$. Due to the ω-deformed Jacobi identity (1.3), we also have

$$c^m_{~li}c^i_{~jk} + c^m_{~ki}c^i_{~lj} + c^m_{~ji}c^i_{~kl} = \delta^m_{~l}\omega_{jk} + \delta^m_{~k}\omega_{lj} + \delta^m_{~j}\omega_{kl},$$

which is equivalent to

$$c^m_{~i[l}c^i_{~jk]} + \delta^m_{~[l}\omega_{jk]} = 0. \tag{2.1}$$

We now find all the orbits of the above defined pair of tensors $(c^k_{~ij}, \omega_{ij})$ under the action of the group $\mathbf{GL}(3,\mathbb{R})$.

We recall that in three dimensions, we have the totally skew symmetric Levi-Civita symbol ϵ_{ijk}, and its totally skew symmetric inverse ϵ^{ijk} such that $\epsilon_{ijk}\epsilon^{ilm} = \delta^l_{~j}\delta^m_{~k} - \delta^m_{~j}\delta^l_{~k}$. This can be used to rewrite the ω-deformed Jacobi identity (2.1). Indeed, since in 3-dimensions every totally skew symmetric 3-tensor is proportional to ϵ_{ijk}, the l.h.s. of (2.1) can be written as

$$t^m = 0 \qquad \text{with} \qquad t^m = (c^m_{~il}c^i_{~jk} + \delta^m_{~l}\omega_{jk})\epsilon^{ljk}.$$

In addidtion, we may use ϵ_{ijk} to write $c^i_{~jk}$ as

$$c^i_{~jk} = n^{il}\epsilon_{jkl} - \delta^i_{~j}a_k + \delta^i_{~k}a_j, \tag{2.2}$$

where the symmetric matrix n^{il} is related to $c^k_{~ij}$ via

$$n^{il} = \tfrac{1}{2}(c^{il} + c^{li}), \qquad \text{with} \qquad c^{il} = \tfrac{1}{2}c^i_{~jk}\epsilon^{jkl}.$$

The vector a_m is related to $c^k_{~ij}$ via

$$a_m = \tfrac{1}{2}\epsilon_{mil}c^{il}.$$

Similarly, we write ω_{ij} as

$$\omega_{ij} = \epsilon_{ijk}b^k, \tag{2.3}$$

with

$$b^k = \tfrac{1}{2}\epsilon^{mik}\omega_{ik}.$$

Thus, in three dimensions the structural constants $(c^k_{~ij}, \omega_{ij})$ of the ω-deformed Lie algebra are uniquely determined via (2.2), (2.3) by specifying a symmetric matrix n^{il} and two vectors a_m and b^k. In terms of the triple (n^{il}, a_m, b^k) the vector t^m is

given by $t^m = 4n^{ml}a_l + 2b^m$, so that the ω-deformed Jacobi identity (2.1) is simply

$$b^i = -2n^{il}a_l. \tag{2.4}$$

Thus, given n^{il} and a_m, the vector b^m defining ω is totally determined. Now we use the action of $\mathbf{GL}(3,\mathbb{R})$ group to bring n^{il} to the diagonal form (it is always possible since n^{il} is symmetric), so that

$$n^{il} = \mathrm{diag}(n^1, n^2, n^3).$$

It is obvious that without loss of generality we always can have

$$n^i = \pm 1, 0 \qquad i = 1, 2, 3.$$

After achiving this we may still use an orthogonal transformation preserving the matrix n^{il} to bring the vector a_m to a simpler form then $a_m = (a_1, a_2, a_3)$. For example in the case $n^{il} = \mathrm{diag}(1,1,1)$ we may always achieve $a_m = (0,0,a)$. Thus to represent a $\mathbf{GL}(3,\mathbb{R})$ orbit of $(c^i{}_{jk}, \omega_{ij})$ it is enough to take n^{il} in the diagonal form with the diagonal elements being equal to $\pm 1, 0$ and to take a_m in the simplest possible form obtainable by the action of $\mathbf{O}(n^{il})$. Finally we notice that the so specified choice of n^{il} is still preserved when the basis is scalled according to

$$e_1 \to \lambda_1 e_1, \qquad e_2 \to \lambda_2 e_2 \qquad e_3 \to \lambda_3 e_3, \tag{2.5}$$

with

$$(\lambda_1\lambda_2 - \lambda_3)n_3 = 0, \qquad (\lambda_3\lambda_1 - \lambda_2)n_2 = 0, \qquad (\lambda_2\lambda_3 - \lambda_1)n_1 = 0.$$

These transformations can be used to scale the vector a_m via

$$a_m \to (\lambda_1 a_1, \lambda_2 a_2, \lambda_3 a_3).$$

We are now in a position to give the full classification of 3-dimensional ω-deformed Lie algebras. In all the types of the classification the commutation relations and the ω are given by:

$$[e_1, e_2] = n^3 e_3 - a_2 e_1 + a_1 e_2, \qquad [e_3, e_1] = n^2 e_2 - a_1 e_3 + a_3 e_1,$$
$$[e_2, e_3] = n^1 e_1 - a_3 e_2 + a_2 e_3$$

$$\omega(e_1, e_2) = -2n^3 a_3, \qquad \omega(e_3, e_1) = -2n^2 a_2, \qquad \omega(e_2, e_3) = -2n^1 a_1.$$

The classification splits into two main branches depending on vanishing or not of a_m.

If $a_m = 0$, then $b^m = 0$ and all the possibilities are given in the following table:

Bianchi type	n^1	n^2	n^3
I	0	0	0
II	1	0	0
VI_0	1	-1	0
VII_0	1	1	0
$VIII$	1	1	-1
IX	1	1	1

$a_m = 0,\ b^m = 0.$

All types from this table have $\omega = 0$ and as such correspond to the usual 3-dimensional Lie algebras.

If $a_m \neq 0$ then, depending on the signature of n^{il}, vector a_m may be spacelike, timelike, null or degenerate. The orthogonal transforamtions we use to normalize this vector preserve its type, so the classification splits according to the causal properties of a_m. If $n^2 = n^3 = 0$ or $n^1 = -n^2 = 1$, $n^3 = 0$, we may use transformations

(2.5) to totally fix a_m. This leads to types V, IV, IV_x and VI_x, VI_y, VI_n below. In all other cases transformations (2.5) can be used to express a_m in terms of only one parameter $a > 0$ so that the different positive parameters a correspond to nonequivalent algebras. The resulting classification is summarized in the following table:

Bianchi type	n^1	n^2	n^3	a_m	b^m
V	0	0	0	(0,0,1)	(0,0,0)
IV	1	0	0	(0,0,1)	(0,0,0)
IV_x	1	0	0	(1,0,0)	(-2,0,0)
VI_a	1	-1	0	$(0,0,a>0)$	(0,0,0)
VI_x	1	-1	0	$(1,0,0)$	(-2,0,0)
VI_y	1	-1	0	$(0,1,0)$	(0,2,0)
VI_n	1	-1	0	$(1,1,0)$	(-2,2,0)
VII_a	1	1	0	$(0,0,a>0)$	(0,0,0)
VII_x	1	1	0	$(1,0,0)$	(-2,0,0)
$VIII_a$	1	1	-1	$(0,0,a>0)$	$(0,0,2a)$
$VIII_{xa}$	1	1	-1	$(a>0,0,0)$	$(-2a,0,0)$
$VIII_{na}$	1	1	-1	$(a>0,0,a)$	$(-2a,0,2a)$
IX_a	1	1	1	$(0,0,a>0)$	$(0,0,-2a)$

In the above two tables all the types which have $b^m = 0$ are just the usual 3-dimensional Lie algebras. Apart from the types I and V all the Bianchi types admit ω deformation. It is interesting to note that types $VIII$ and IX, which in the Lie algebra setting do not admit $a_m \neq 0$ deformation, admit a one-parameter ω-deformations.

We have the following theorem.

Theorem 2.1. *All the 3-dimensional ω-deformed Lie algebras are given in the following table*

Bianchi type	n^1	n^2	n^3	(a_1, a_2, a_3)	(b^1, b^2, b^3)
IV_x	*1*	*0*	*0*	*(1,0,0)*	*(-2,0,0)*
VI_x	*1*	*-1*	*0*	$(1,0,0)$	*(-2,0,0)*
VI_y	*1*	*-1*	*0*	$(0,1,0)$	*(0,2,0)*
VI_n	*1*	*-1*	*0*	$(1,1,0)$	*(-2,2,0)*
VII_x	*1*	*1*	*0*	$(1,0,0)$	*(-2,0,0)*
$VIII_a$	*1*	*1*	*-1*	$(0,0,a>0)$	$(0,0,2a)$
$VIII_{xa}$	*1*	*1*	*-1*	$(a>0,0,0)$	$(-2a,0,0)$
$VIII_{na}$	*1*	*1*	*-1*	$(a>0,0,a)$	$(-2a,0,2a)$
IX_a	*1*	*1*	*1*	$(0,0,a>0)$	$(0,0,-2a)$

They satisfy the commutation relations

$$[e_1, e_2] = n^3 e_3 - a_2 e_1 + a_1 e_2, \qquad [e_3, e_1] = n^2 e_2 - a_1 e_3 + a_3 e_1,$$
$$[e_2, e_3] = n^1 e_1 - a_3 e_2 + a_2 e_3$$

$$\omega(e_1, e_2) = -2n^3 a_3, \qquad \omega(e_3, e_1) = -2n^2 a_2, \qquad \omega(e_2, e_3) = -2n^1 a_1.$$

with the real parameters $(n^1, n^2, n^3, a_1, a_2, a_3)$ specified in the table. Algebras corresponding to different $(n^1, n^2, n^3, a_1, a_2, a_3)$ are nonequivalent.

Finally we show that any ω-deformed Lie algebra must have quite nontrivial structure constants. Indeed, in any dimension $\dim V = n > 2$ the structure constants of an ω-deformed Lie algebra, which are defined by $[e_i, e_j] = c^k{}_{ij} e_k$, may be

decomposed as follows:

$$c^i{}_{jk} = \alpha^i{}_{jk} + a_k\delta^i{}_j - a_j\delta^i{}_k,$$

where

$$\alpha^i{}_{ik} = 0, \qquad a_k = \tfrac{1}{n-1}c^i{}_{ik}.$$

Then a simple calculation using the ω-deformed Jacobi identity (1.3) shows that

$$\omega(e_j, e_k) = \tfrac{n-1}{n-2}a_i\alpha^i{}_{jk}.$$

This shows that nonvanishing ω is only possible if both a_i and $\alpha^i{}_{jk}$ are nonvanishing.

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